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SELFSIMILAR PROBLEMS OF STAR-SHAPED CRACK DEVELOPMENT UNDER THE ACTION OF A CLEAVING GAS FLOW*

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The problem of the development of a system of a system of radial slits from zero at a constant rate under plane strain conditions is considered. For instance, this problem occurs in describing fracturing of a stratum by hydraulic rupture cracks. Crack cleavage can be produced by a gas being formed during the explosion of a cylindrical charge /1/ or by special powder pressure generators pressed into a borehole /2/.

Selfsimilar solutions of problems on the development of normal separation crack stars were obtained by the Smirnov-Sobolev method of functionally invariant solutions /3, 4/. The solution of the problem of the cleavage of a star-shaped crack by an isothermal inviscid gas flow being liberated instantaneously at the initial time is presented.

Selfsimilar problems on the development of a star-shaped crack under antiplane strain conditions were examined in /5, 6/. Investigation of the cleaving action of a viscous isothermal gas flow on rectilinear isolated slit propagation is performed in /7/, where it is shown that a deceleration shock is formed in the gas flow under certain governing parameters of the problem.

1. Formulation of the problem. In a supporting xy plane, let a system of $2n$ radial slits distributed uniformly over an angle start to be developed at the initial time from the origin of coordinates with a constant velocity v. A normal load P is applied to the slit edges.

The elastic displacements of the medium are described by the equations of dynamic elasticity theory /3, 4/

$$w_i = u_i + v_i, \quad \Delta u_i = \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}u_i, \quad \Delta v_i = \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}v_i; \quad i = 1, 2 \tag{1.1}$$
$$\frac{\partial}{\partial y}u_1 = \frac{\partial}{\partial x}u_2, \quad \frac{\partial}{\partial x}v_1 = -\frac{\partial}{\partial y}v_2 \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Here $u_i(x, y; t)$, $v_i(x, y; t)$ are potential and solenoidal components of the displacement vector $w_i(x, y; t)$, and c_1, c_2 are longitudinal and transverse wave velocities $(c_1 > c_2)$.

The stress tensor components are related to the displacements by Hooke's law (μ is the Lamé constant)

$$\sigma_{xx} = \mu\left[\left(\frac{c_1}{c_2}\right)^2 \operatorname{div}\mathbf{w} - 2\frac{\partial}{\partial y}w_2\right] \tag{1.2}$$
$$\sigma_{yy} = \mu\left[\left(\frac{c_1}{c_2}\right)^2 \operatorname{div}\mathbf{w} - 2\frac{\partial}{\partial x}w_1\right], \quad \sigma_{xy} = \mu\left[\frac{\partial}{\partial x}w_2 + \frac{\partial}{\partial y}w_1\right]$$

We will specify the boundary conditions on the $2n$ slit edges for (1.1) and (1.2) in the form

$$\sigma_{nn} = -P(r, \varphi_0), \ \sigma_{nr} = 0; \ \varphi_0 = k\pi/n, \ r < vt \tag{1.3}$$
$$w_n = 0, \ \sigma_{n\tau} = 0; \ \varphi_0 = k\pi/n, \ vt < r < c_1 t$$
$$r = (x^2 + y^2)^{1/2}, \ \varphi = \operatorname{arctg}(y/x); \ k = 0, \ldots, 2n-1$$

It is assumed that the normal stress tensor component on the crack tips has the root singularity

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$$\sigma_{nn}(r, \varphi_0; t) \underset{r \to vt}{\longrightarrow} \frac{K_I(t)}{\sqrt{2\pi(r - vt)}} \tag{1.4}$$

Here $K_I(t)$ is the stress intensity factor (SIF).

Because of the symmetry of Problem (1.1)-(1.4) it is sufficient to consider the domain bounded by the surfaces $\varphi = 0$ and $\varphi = \pi/n$ (Fig.1). The unit vector normal to the boundary of this angular domain has the components $(n_x, n_y) = (-1)^{1-n\varphi_0/\pi} \times [-\sin\varphi_0, \cos\varphi_0]$ ($\varphi_0 = 0, \pi/n$). The normal component of the displacement vector and the normal and tangential components of the stress vector on the surface of the angular domain can be represented in the form

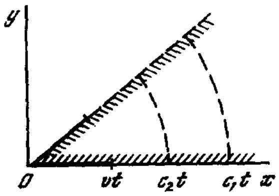

Fig.1

$$w_n(r, \varphi_0, t) = (-1)^{n\varphi_0/\pi} \cdot (w_2\cos\varphi_0 - w_1\sin\varphi_0) \tag{1.5}$$

$$\sigma_{nn}(r, \varphi_0, t) = \sigma_{xx}\sin^2\varphi_0 + \sigma_{yy}\cos^2\varphi_0 - \sigma_{xy}\sin 2\varphi_0$$

$$\sigma_{n\tau}(r, \varphi_0, t) = \sigma_{xy}\cos 2\varphi_0 - \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\varphi_0$$

$$\varphi_0 = 0, \ \pi/n$$

2. *Solution of the selfsimilar problem.* We will examine the case of selfsimilar loading

$$P(r, \varphi_0, t) = P_0(t/t_0)^{l-1}P(\xi), \ \xi = r/(vt), \ \varphi_0 = 0, \ \pi/n \tag{2.1}$$

where P_0, t_0 are pressure and time dimensionality constants, and l is a non-negative integer.

We will use the Smirnov-Sobolev method of functionally invariant solutions to solve Problem (1.1)-(1.4) under the loading (2.1). We introduce homogeneous functions in the variables x, y, t /3, 4/

$$U_i = \partial^l u_i/\partial t^l, \ V_i = \partial^l v_i/\partial t^l; \ i = 1, 2 \tag{2.2}$$

The functions U_i satisfy the wave equation for the longitudinal waves and V_i for the transverse waves.

We represent U_i and V_i as real parts of analytic functions of the complex variables z_1 and z_2

$$U_i(x/(c_1t), y/(c_1t)) = \operatorname{Re} U_i^l(z_1), \ V_i(x/(c_1t), y/(c_1t)) = \operatorname{Re} V_i^l(z_2); \ i = 1, 2 \tag{2.3}$$

$$z_k^{-1} = c_k^{-1}\operatorname{ch}\{n(\beta_k - i\varphi)\}, \ \beta_k = \operatorname{arch}\xi_k^{-1}, \ \xi_k = r/(c_kt) \tag{2.4}$$

The transformation (2.4) maps the angular domain presented in Fig.1 into the upper half-plane ($z_k = x_k + iy_k$, $y_k \geqslant 0$; $k = 1, 2$).

Substituting (2.3) into (1.2) and taking (1.1) into account, we obtain for the stress

$$\overset{\circ}{\sigma}_{xy} = \mu\operatorname{Re}\left\{2(U_2^l)'\frac{\partial z_1}{\partial x} + (1 - \omega_2^2)(V_2^l)'\frac{\partial z_2}{\partial x}\right\} \tag{2.5}$$

$$\overset{\circ}{\sigma}_{xx} = \mu\operatorname{Re}\left\{\left[\left(\frac{c_1}{c_2}\right)^2(1 + \omega_1^{-2}) - 2\right](U_2^l)'\frac{\partial z_1}{\partial y} - 2(V_2^l)'\frac{\partial z_2}{\partial y}\right\}$$

$$\overset{\circ}{\sigma}_{yy} = \mu\operatorname{Re}\left\{\left[\left(\frac{c_1}{c_2}\right)^2(1 + \omega_1^{-2}) - 2\omega_1^{-2}\right](U_2)'\frac{\partial z_1}{\partial y} + 2(V_2^l)'\frac{\partial z_2}{\partial y}\right\}$$

$$(U_1^l)' = \omega_1^{-1}(U_2^l)', \quad (V_1^l)' = -\omega_2(V_2^l)'; \quad \omega_k = \frac{\partial z_k}{\partial y}\Big/\frac{\partial z_k}{\partial x}, \quad k = 1, 2$$

Differentiating (2.4) we find

$$\frac{\partial}{\partial x}z_k = \Omega_k(x - iy\sqrt{1 - \xi_k^2}), \quad \frac{\partial}{\partial y}z_k = \Omega_k(y + ix\sqrt{1 - \xi_k^2}) \tag{2.6}$$

$$\frac{\partial}{\partial t}z_k = -\Omega_k r^2/t$$

$$\Omega_k = \frac{n}{r^2}z_k\frac{1 - \theta_k^{2n}}{1 + \theta_k^{2n}}(1 - \xi_k^2)^{-1/2}, \quad \theta_k = \rho_k e^{i\varphi}, \quad \rho_k = \frac{1 - \sqrt{1 - \xi_k^2}}{\xi_k}$$

We introduce the analytic functions $W_1'(z_1)$ and $W_2'(z_2)$ such that on the boundary of the angular domain ($\varphi = \varphi_0$; $\varphi_0 = 0, \pi/n$)

$$dU_2^l/dz_1 = J_1(2 - \xi_2^2)\,\omega_0(\cos\varphi_0\sqrt{1 - \xi_1^2} - i\sin\varphi_0)\,W_1'(z_1) \tag{2.7}$$

$$dV_2^l/dz_2 = -2J_2\sqrt{1-\xi_2^2}\,(\cos\varphi_0 - i\sin\varphi_0\sqrt{1-\xi_2^2})\,W_2'(z_2)$$

$$J_k^{-1} = \sqrt{1-\xi_k^2}\left[(2-\xi_2^2)\,\omega_0 - 2\frac{dz_2(\varphi_0)}{dz_1(\varphi_0)}\right],\quad \omega_0 = \frac{\partial z_2}{\partial t}\Big/\frac{\partial z_1}{\partial t}$$

It follows from (1.5), (2.5) and (2.7) that the boundary condition $\sigma_{n\tau}(\varphi=\varphi_0)=0$ is satisfied if for $\varphi=\varphi_0$

$$W_1'(z_1) = W_2'(z_2) \tag{2.8}$$

We find from the definition (2.4) of the complex variables z_1 and z_2 that for $\varphi=\varphi_0$

$$\frac{dz_2(\varphi=\varphi_0)}{dz_1(\varphi=\varphi_0)} = \frac{c_1}{c_2}\frac{z_2}{z_1}\sqrt{\frac{c_2^2-z_2^2}{c_1^2-z_1^2}}\sqrt{\frac{1-\xi_1^2}{1-\xi_2^2}}$$

On the boundary of the angular domain $\varphi=\varphi_0$ we have from (1.5), (2.5), (2.7) and (2.8)

$$\begin{aligned}
&\frac{\partial^l}{\partial t^l}\sigma_{nn} = \mu\,\mathrm{Im}\left\{\frac{(2-\xi_2^2)^2 - 4\sqrt{1-\xi_1^2}\sqrt{1-\xi_2^2}}{c_2\xi_2\sqrt{1-\xi_1^2}\,[(2-\xi_2^2)\,\omega_0 - 2\,dz_2(\varphi_0)/dz_1(\varphi_0)]}\times\right.\\
&\qquad\left.\frac{\partial z_2}{\partial t}W_1'(z_1)\right\}\\
&\partial^l w_n/\partial t^l = \mathrm{Re}\,W_1
\end{aligned} \tag{2.9}$$

We formulate the boundary value problem for the function W_1' in the upper half-plane $z = z_1/c_1$. We have from (1.3) and (2.9)

$$\begin{aligned}
&\mathrm{Im}\,z = 0,\; z_* < |\mathrm{Re}\,z| < 1,\; \mathrm{Re}\,W_1' = 0\\
&\mathrm{Im}\,z = 0,\; z_v < |\mathrm{Re}\,z| < z_*,\; \mathrm{Re}\,W_1' = 0\\
&\mathrm{Im}\,z = 0,\; |\mathrm{Re}\,z| < z_v,\; \mathrm{Im}\,W_1' =\\
&\quad -\frac{tv}{n\mu z_1}\frac{(1-m_1^2\xi^2)\xi}{\sqrt{1-(z_1/c_1)^2}}\frac{[2\Lambda_2 - (2-m_2^2\xi^2)\,\Lambda_1]}{[(2-m_2^2\xi^2)^2 - 4\sqrt{1-m_1^2\xi^2}\sqrt{1-m_2^2\xi^2}]}\frac{\partial^l}{\partial t^l}p\\
&z_*^{-1} = \mathrm{ch}\,\{n\,\mathrm{arch}\,(c_1/c_2)\},\; z_v^{-1} = \mathrm{ch}\,\{n\,\mathrm{arch}\,(c_1/v)\},\; m_k = v/c_k\\
&\Lambda_k = \frac{(m_k\xi)^{2n} + \left[1-\sqrt{1-m_k^2\xi^2}\right]^{2n}}{(m_k\xi)^{2n} - \left[1-\sqrt{1-m_k^2\xi^2}\right]^{2n}}\,\mathrm{th}\,\{n\,\mathrm{arch}\,(m_k^{-1}\xi^{-1})\},\quad k = 1, 2
\end{aligned} \tag{2.10}$$

Therefore, Problem (1.1)-(1.4) reduces to a mixed Keldysh-Sedov problem for the function W_1' whose solution can be represented in the form /3/

$$W_1'\left(\frac{z_1}{c_1}\right) = \frac{1}{(z^2-z_v^2)^{l+1/2}}\left\{\frac{1}{\pi}\int\limits_{-z_v}^{z_v}\frac{(s^2-z_v^2)^{l+1/2}\,\mathrm{Im}\,W_1'\,ds}{s-z} + iz\sum_{j=0}^{l-1}A_j z^{2j}\right\} \tag{2.11}$$

The solution (2.12) contains l constants $A_j\;(j = 0,\ldots, l-1)$ whose values are determined from the system

$$\frac{\partial^j}{\partial t^j}\sigma_{nn}(vt-0,\varphi_0,t) = -\frac{\partial^j}{\partial t^j}P(vt,t);\quad j = 0,\ldots, l-1$$

The specific form of the load (2.1) is due to the physical formulation of the problem. In the special case of a loading concentrated at the origin the SIF and the constants A_j can be determined from the algorithm proposed in /3/.

We consider below the problem of crack star development under the effect of a cleaving gas flow.

3. ***The problem of a point source.*** At the initial time let a mass M of gas be liberated during the detonation of a cylindrical charge and let a star-shaped crack be formed, which will propagate into an elastic mass if under the action of a cleaving gas flow. We will confine ourselves to considering the initial phase of fracturing of the medium when the influence of gas viscosity can be neglected. It is established experimentally that for high detonation product pressures on the crack edges its propagation from the borehold during a relatively prolonged time interval will occur at a constant velocity /8/.

The motion of an isothermal inviscid gas in a crack is described by the gas conservation of mass and momentum equations

$$\frac{\partial}{\partial t}(\rho w_n) + \frac{\partial}{\partial r}(\rho u w_n) = 0, \quad \rho \frac{d}{dt} u + \frac{\partial}{\partial r} P = 0, \quad P = c^2 \rho \tag{3.1}$$

where P, ρ, u are the gas pressure, density and velocity in a crack and c is the isothermal velocity of sound.

We will consider the development of the crack in a slightly permeable mass if when the gas leakage through the wall can be neglected. The total gas mass M in the crack should be conserved Here

$$M = 2n \int_0^{L(t)} dr w_n \rho, \quad L(t) = vt \tag{3.2}$$

By virtue of the symmetry of the problem and the absence of a gas source at the origin, we have

$$u(r = 0, t) = 0 \tag{3.3}$$

Neglecting the counter-pressure of the background gas we obtain that Problem (3.1)-(3.3) is selfsimilar in the variable $\xi = r/(vt)$. **Integrating** (3.1) taking (3.2) and (3.3) into account, we obtain

$$u = v\xi\delta_0(\xi_0 - \xi), \; P = P_0\left(\frac{t}{t_0}\right)\delta_0(\xi_0 - \xi) \tag{3.4}$$
$$(\varphi = k\pi/n, \; k = 0, \ldots, 2n - 1)$$

where $\delta_0(x)$ is the Heaviside function and $\xi_0 = c/v$ is the coordinate of the front of the dissipating gas.

Since the crack propagates in impermeable rock, we have $\xi_0 = 1$ for $c \gg v$.

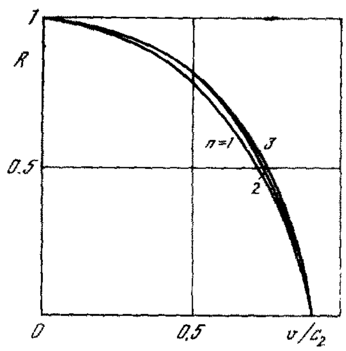

Fig.2

Comparing (3.4) and (2.1), we obtain

$$l = 0, \; w_n = w_0 w(\xi) \tag{3.5}$$

where w_0 is the crack characteristic width, that will be determined below.

The problem of the development of the crack is linear in the gas pressure (3.4) applied to its edges. Consequently, the crack opening $w_n(\xi)$ and the SIF can be represented in the form

$$w_n(\xi) = \int_0^{\xi_0} d\zeta w_*(\xi, \zeta), \quad K_I(t) = \int_0^{\xi_0} d\zeta K_I^*(t, \zeta) \tag{3.6}$$

where $w_*(\xi, \zeta)$ and $K_I^*(t, \zeta)$ are the crack opening and the SIF that correspond to the load ($\delta_1(x)$ is the delta function)

$$P(\xi, t) = P_0(t/t_0)\delta_1(\xi - \zeta); \; \varphi = k\pi/n, \; k = 0, \ldots, 2n - 1 \tag{3.7}$$

Finding the potential W_1' for the load (3.7) from (2.12), we obtain the crack profile from the second formula in (2.9)

$$w_*(\xi, \zeta) = w_0 Q(\zeta) \ln \frac{\sqrt{z_v^2 - z_\zeta^2} + \sqrt{z_v^2 - z^2}}{\sqrt{|z^2 - z_\zeta^2|}} \tag{3.8}$$

$$Q(\zeta) = \frac{\sqrt{1 - m_1^2\zeta^2}\,[(2 - m_2^2\zeta^2)\,\Lambda_1 - 2\Lambda_2]}{(2 - m_2^2\zeta^2)^2 - 4\sqrt{1 - m_1^2\zeta^2}\,\sqrt{1 - m_2^2\zeta^2}}$$
$$z^{-1} = \operatorname{ch}\{n \operatorname{arch}(m_1^{-1}\xi^{-1})\},\quad z_\zeta^{-1} = \operatorname{ch}\{n \operatorname{arch}(m_1^{-1}\zeta^{-1})\}$$
$$w_0 = 2P_0 v\,(\pi\mu)^{-1}$$

By using (1.4) we determine from the first formula in (2.9) and (2.11)

$$K_I^*(t, \zeta) = 2P_0 t_0 \sqrt{v/(\pi t n)}\, R(n, m_1, m_2; \zeta) \tag{3.9}$$
$$R(n, m_1, m_2; \zeta) = \frac{z_v}{\sqrt{z_v^2 - z_\zeta^2}}\, \frac{Q(\zeta)}{Q(1)} \left(\frac{1 - z_v^2}{1 - m_1^2}\right)^{1/4}$$

In the special case of a load concentrated at the origin $(\zeta \to 0)$ the SIF (3.9) has the form

$$R(n, m_1, m_2; 0) = \frac{1}{2}\, \frac{m_2^2}{(m_2^2 - m_1^2)\, Q(1)} \left(\frac{1 - z_v^2}{1 - m_1^2}\right)^{1/4} \tag{3.10}$$

For a rectilinear slit $(n = 1)$ the quantity $K_I(t)$ is evaluated in /3, 4/. For $n = 1$ we have from (3.10)

$$R(1, m_1, m_2; 0) = \frac{(2 - m_2^2)^2 - 4\sqrt{1 - m_1^2}\,\sqrt{1 - m_2^2}}{2m_1^2\,[1 - (m_2/m_1)^2]\,\sqrt{1 - m_1^2}}$$

The dependence of the SIF $R(n, m_1, m_2; 0)$ on the rate of crack propagation is presented in Fig.2 for $c_2/c_1 = 0.6$.

We determine the constant $P_0 t_0$ from the conservation law (3.2) for the gas mass M

$$P_0 t_0 = \frac{c}{2v} \left[\frac{\pi\mu M w_0}{n} \left(\int_0^{\xi_0} d\xi\, w_n(\xi)\right)^{-1}\right]^{1/2} \tag{3.11}$$

Formulas (3.4), (3.6), (3.8), (3.9) and (3.11) yield the complete solution of the problem of star-shaped crack propagation under the effect of a gas mass M being liberated instantaneously at the initial time.

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